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Polymer Statistics and Fermionic Vector Models¹

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Abstract

We consider a variation of $O(N)$ -symmetric vector models in which the vector components are Grassmann numbers. We show that these theories generate the same sort of random polymer models as the $O(N)$ vector models and that they lie in the same universality class in the large- N limit. We explicitly construct the double-scaling limit of the theory and show that the genus expansion is an alternating Borel summable series that otherwise coincides with the topological expansion of the bosonic models. We also show how the fermionic nature of these models leads to an explicit solution even at finite- N for the generating functions of the number of random polymer configurations.

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The statistical mechanics of randomly branching polymers (sometimes called discrete filamentary surfaces) is of interest in condensed matter physics where the connected polymer chains can be thought of as describing molecules. Random polymers have also been of interest as examples of random geometry systems which in many cases are exactly solvable and which have been argued to represent a certain dimensionally-reduced phase of Polyakov string theory in target space dimensions $D > 1$ [1, 2]. They therefore serve as toy models for more complicated higher dimensional problems such as statistical models of discretized surfaces which have been studied in the context of string theory and lower dimensional quantum gravity (see [3] for a review). $O(N)$ -symmetric vector field theories [4] provide non-perturbative models of randomly branching chains where the order in the $1/N$ expansion coincides with the genus, or number of loops in the molecules of the ensemble. These models exhibit phase transitions in the large- N limit at which infinitely long polymers dominate the statistical sum and a continuum limit analogous to that found in the matrix model representation of random surface theories is reached [5]–[9]. The simplicity of the random polymers as compared to random surfaces allows a more explicit solution, even in dimensions $D > 1$, where some of the ideas about scaling and other critical behaviour can be tested. These simpler polymer structures are reflected in the linearity of the vector field theories in contrast to the non-linearity of matrix ones. The multicritical series generated by these models give generalized statistical systems which interpolate between the Cayley tree at one end and the ordinary random walk at the other [1]. Some supersymmetric generalizations of the $O(N)$ vector model have been studied in [10].

In this Letter we shall show that a vector theory with purely fermionic degrees of freedom can also be used to represent random polymers. This model exhibits the same critical behaviour in the large- N limit as the $O(N)$ vector model, but it has a structure which is in some respects simpler. In particular, we will show that its genus expansion is an alternating series which is Borel summable and that it represents a rare example of a random geometry theory whose explicit solution can be written down even at finite N . It therefore provides an explicit, well-defined generating function for a given number of polymer configurations in a random polymer system. It can also be combined with other polymer models to generate new types of generating functions for random surface theories. For example, it can be combined with the $O(N)$ vector model to study random polymer theories with either only even or only odd genera.

Let us start by reviewing some features of the $O(N)$ vector model with partition function [6, 7, 9]

$$Z_S(t, g; N) = \int \prod_{i=1}^N d\phi_i \exp \left\{ -t \sum_{i=1}^N \phi_i^2 + \frac{g}{N} \left(\sum_{i=1}^N \phi_i^2 \right)^2 \right\} \quad (1)$$

where the integration is over \mathbb{R}^N . The model (1) is invariant under the orthogonal transfor-

mation $\phi_i \rightarrow \sum_{j=1}^N S_{ij} \phi_j$, $S \in O(N)$. This symmetry restricts the observables of the theory to those which are functions of $\phi^2 \equiv \sum_i \phi_i^2$. When the coupling constants t and g are positive, it is straightforward to show that the formal expression (1) counts the number of randomly branching polymers, both those with a tree-like structure and those with arbitrarily many loops. The fact that the integral is divergent is a reflection of the divergence of the statistical sum.

The statistical sum over polymers coincides with the expansion of the free energy

$$F_S \equiv -\frac{1}{N} \log \left(\frac{Z_S(t, g; N)}{Z_S(t, 0; N)} \right) \quad (2)$$

in Feynman diagrams. The propagator is

$$\langle \phi_i \phi_j \rangle_S \equiv \frac{\int \prod_k d\phi_k \phi_i \phi_j e^{-t\phi^2}}{\int \prod_k d\phi_k e^{-t\phi^2}} = \delta_{ij} \quad , \quad (3)$$

the vertex is

$$\langle \phi_i \phi_j \phi_k \phi_l \rangle_S = \frac{1}{(2t)^2} (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}) \quad (4)$$

and the free energy is the sum of all connected diagrams with four-point couplings (Figs. 1 and 2). The dual graphs to these Feynman diagrams, defined by associating a vertex (molecule) in the center of each of the scalar loops and lines (bonds) connecting vertices by crossing each of the Feynman 4-point couplings, are the random walk diagrams shown in Fig. 2. The number of molecules n appears as the power in N^{n-1} while the number of bonds b is given by the power in $(g/Nt^2)^b$ [5, 6]. This identifies $\frac{1}{N} = e^\mu$ with the fugacity μ of the polymer and $\frac{g}{t^2} = e^{-L}$ with the length L of the branched chain (the usual action terms for a random walk model) [1]. The vector model partition function is therefore the generating function for the number of polymer configurations with b bonds and $\ell = b - n + 1$ loops. Note that this sum includes the self-bonding polymers which are generated by Wick contracting several propagators into single loops (as opposed to multi-loops) and occur in the expansion only for $\ell \geq 1$ (Fig. 2).

$$\begin{array}{c} i \qquad j \\ \hline \end{array} = \delta_{ij} \qquad \begin{array}{c} i \qquad j \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ l \qquad k \end{array} = \frac{1}{4t^2} (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl})$$

Figure 1: Feynman rules for the $O(N)$ vector model.

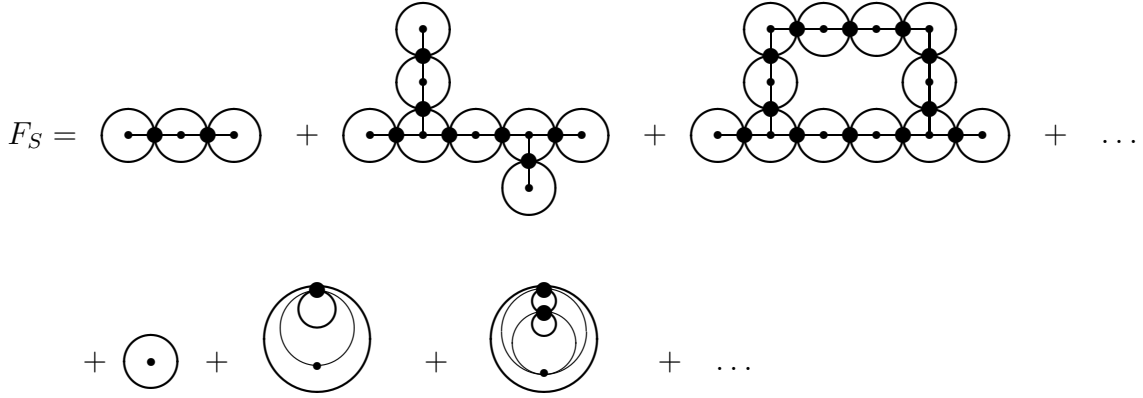


Figure 2: Feynman diagram expansion of the $D = 0$ quantum field theory (1). The thick lines and vertices represent the Feynman graphs, while the thin lines and vertices represent the dual graphs which form polymer networks (or random walks). The first set of diagrams represent those types of branching polymers associated with the higher orders of N in vertex contractions such as $\langle (\phi^2)^2 \rangle_S = (N^2 + 2N)/4t^2$ in Fig. 1 (so that $n > 1$ and bonds connect different molecules), while the second set of diagrams representing the self-bonding polymers are associated with the lower powers of N (so that $\ell = b$).

From an analytic point of view, the perturbative expansion of the partition function

$$\frac{Z_S(t, g; N)}{Z_S(t, 0; N)} = \sum_{n \geq 0} \frac{1}{n!} \left(\frac{g}{N} \right)^n \left\langle (\phi^2)^{2n} \right\rangle_S \quad (5)$$

is completely determined by the Gaussian moments

$$\left\langle (\phi^2)^{2n} \right\rangle_S \equiv \frac{\int \prod_i d\phi_i (\phi^2)^{2n} e^{-t\phi^2}}{\int \prod_i d\phi_i e^{-t\phi^2}} = t^{N/2} \left(\frac{\partial^2}{\partial t^2} \right)^n t^{-N/2} \quad (6)$$

as

$$\frac{Z_S(t, g; N)}{Z_S(t, 0; N)} = \sum_{n=0}^{N_\Lambda} \frac{(N + 4n - 2)!!}{2^{2n} n! (N - 2)!!} \left(\frac{g}{Nt^2} \right)^n \quad (7)$$

Here we have introduced an “ultraviolet” cutoff $N_\Lambda \in \mathbb{Z}^+$ to make the partition function well-defined. In the limit $N_\Lambda \rightarrow \infty$, the series (7) is a non-Borel summable asymptotic series reflecting the divergence of the original integral and also the divergence of the statistical sum. As is familiar from the study of random surfaces [3], even though the series is divergent, if arranged as a power series in $1/N$, rather than g , the terms in this series are individually convergent and it is the sum over genera ℓ which is asymptotic [1],[5]–[9]. With the cutoff N_Λ in (7), the partition function is an analytic function of N , but it is only well-defined at $N = \infty$ when this ultraviolet cutoff is removed.

The large N expansion is a saddle point computation of the integral (1). After integration over angular variables, the partition function (1) can be written in terms of the radial

coordinate in Euclidean N -space as

$$Z_S(t, g; N) = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_0^\infty d\phi \phi^{N-1} e^{-t\phi^2 + \frac{g}{N}\phi^4} \quad (8)$$

In the infinite- N limit, which counts the tree-graphs ($\ell = 0$), the integral (8) can be evaluated using the saddle-point approximation. Rescaling $\phi \rightarrow \phi/\sqrt{N}$ the stationary condition for the effective action $-Nt\phi^2 + gN\phi^4 + N \log \phi$ in (8) is

$$2t\phi^2 - 4g\phi^4 = 1 \quad (9)$$

The solution of (9) which is regular at $g = 0$ and which minimizes the effective action is

$$\phi_0^2 = \frac{t}{4g} \left(1 - \sqrt{1 - \frac{4g}{t^2}} \right) \quad (10)$$

The tree-level free energy is then the value of the effective action in (8) evaluated at the saddle-point (10),

$$F_S^{(0)} \equiv \lim_{N \rightarrow \infty} -\frac{1}{N} \log Z_S(t, g; N) = \frac{1}{2} \left\{ \frac{1}{2} + \frac{t}{4g} \left(t - \sqrt{t^2 - 4g} \right) - \log \left(\frac{t}{4g} - \frac{1}{4g} \sqrt{t^2 - 4g} \right) \right\} \quad (11)$$

This free energy becomes non-analytic at the critical point $g = g_c \equiv t^2/4$ where the 2 solutions of the quadratic equation (9) coalesce. There the minimum of the effective action in (8) disappears and it becomes unbounded [7, 8], so that the saddle point solution is no longer valid. There is a second order phase transition at the coupling $g = g_c$ with susceptibility exponent $\gamma^{(0)} = \frac{1}{2}$ [1, 5, 6]. This critical point is identified as the “continuum” limit of the random polymer theory where the number of branches, and hence the lengths, of the tree-graphs becomes infinite. The higher-loop contributions (molecular networks) can be found in [9] and their “continuum” limit is associated with an infinite number of molecules, thus tracing out a continuum filamentary surface [1, 5].

Random polymer models can also be generated by the fermionic vector model with partition function

$$Z_F(t, g; N) = \int d\psi d\bar{\psi} e^{t\bar{\psi}\psi - \frac{g}{2N}(\bar{\psi}\psi)^2} \quad (12)$$

where $\psi_i, \bar{\psi}_i, i = 1, \dots, N$, are independent anticommuting nilpotent variables, $\bar{\psi}\psi \equiv \sum_{i=1}^N \bar{\psi}_i \psi_i$ and the integration measure $d\psi d\bar{\psi} \equiv \prod_{i=1}^N d\psi_i d\bar{\psi}_i$ is defined using the usual Berezin rules for integrating Grassmann variables, $\int d\psi_i \psi_i = 1, \int d\psi_i 1 = 0$. The model (12) possesses a continuous symmetry, $\psi_i \rightarrow \sum_j U_{ij} \psi_j, \bar{\psi}_i \rightarrow \sum_j \bar{\psi}_j (U^{-1})_{ji}$ with $U \in GL(N, \mathbb{C})$. There is a further discrete symmetry under the “chiral” transformation $\psi_i \rightarrow \bar{\psi}_i, \bar{\psi}_i \rightarrow -\psi_i$ for any i , which is the analog of the reflection symmetry $\phi_i \rightarrow -\phi_i$ of the $O(N)$ vector model. Together, these two symmetries restrict the observables of the model to those which are functions only of

$\bar{\psi}\psi$. We shall now show that the model (12) possesses a random geometry interpretation and critical behaviour similar to that of the $O(N)$ vector model. A main difference with the $O(N)$ vector model is that the integration over Grassmann variables in the generating function (12) for the polymers is a well-defined finite polynomial in the coupling constants g and t because nilpotency of the components ψ_i and $\bar{\psi}_i$ implies that $(\bar{\psi}\psi)^{N+1} = 0$. The dimension N itself provides a cutoff on the number of terms (and polymers) in the Feynman diagram expansion of (12). Here, rather than making the partition function integration well-defined as in the bosonic case, the large- N limit is needed to generate the full ensemble of randomly-branched chains.

The Feynman diagrams for the fermion vector theory (12) have propagator

$$\langle \bar{\psi}_i \psi_j \rangle_F \equiv \frac{\int d\psi d\bar{\psi} \bar{\psi}_i \psi_j e^{t\bar{\psi}\psi}}{\int d\psi d\bar{\psi} e^{t\bar{\psi}\psi}} = \delta_{ij} \quad (13)$$

and the four-Fermi interaction vertex is

$$\langle \bar{\psi}_i \psi_j \bar{\psi}_k \psi_l \rangle_F = \frac{1}{t^2} (\delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk}) \quad (14)$$

These Feynman rules have the same graphical representation shown in Fig. 1 with a left-handed orientation for the lines (ingoing lines into a vertex for $\bar{\psi}_i$ components and outgoing lines for ψ_j components). Now the graphs are formed from all connected 4-point diagrams which preserve this orientation. The Feynman rules also associate a factor of -1 to each fermion loop in a Feynman graph.

The perturbative expansion of (12)

$$\frac{Z_F(t, g; N)}{Z_F(t, 0; N)} = \sum_{n \geq 0} \frac{(-1)^n}{n!} \left(\frac{g}{2N} \right)^n \langle (\bar{\psi}\psi)^{2n} \rangle_F \quad (15)$$

is completely determined by the normalized Gaussian moments

$$\langle (\bar{\psi}\psi)^{2k} \rangle_F \equiv \frac{\int d\psi d\bar{\psi} (\bar{\psi}\psi)^{2k} e^{t\bar{\psi}\psi}}{\int d\psi d\bar{\psi} e^{t\bar{\psi}\psi}} = t^{-N} \left(\frac{\partial^2}{\partial t^2} \right)^k t^N \quad (16)$$

to be

$$\frac{Z_F(t, g; N)}{Z_F(t, 0; N)} = \sum_{n=0}^{N_2} (-1)^n \frac{N!}{n!(N-2n)!} \left(\frac{g}{2Nt^2} \right)^n \quad (17)$$

where $N_2 = N/2$ (respectively $(N-1)/2$) when N is even (odd). The perturbation series is a finite sum which represents the same sort of random walk distribution as shown in Fig. 2 except that it only includes polymers with up to N_2 bonds. From a diagrammatic point of view, the alternating nature of the series arises from the minus sign associated with fermion loops. Term by term, this series can be made identical with the terms of same order in g in (7) by the analytical continuation $N \rightarrow -N/2$ in (17) (so that $N_\Lambda = N_2$ in (7)). The factor of

2 is associated with the doubling of degrees of freedom in the fermionic case. Thus, after the substitution $N \rightarrow N/2$, the large N expansion of the fermionic vector model is identical to that of the $O(N)$ vector model *except* that it is an alternating series in $1/N$. The coefficient of $1/N^k$ in the former and $1/(-N)^k$ in the latter are identical. Now, however, the alternating nature of the fermionic vector series makes its $N \rightarrow \infty$ limit Borel summable, and as such it defines a better behaved statistical theory.

To explicitly carry out the $\frac{1}{N}$ -expansion of the fermionic vector model, we introduce a scalar Hubbard-Stratonovich field φ by inserting the identity $1 = \int d\varphi e^{-\frac{g}{2N}(\varphi + i\bar{\psi}\psi)^2}$ into the partition function integral (12) to write it as

$$Z_F(t, g; N) = \int d\varphi e^{-g\varphi^2/2N} \int d\psi d\bar{\psi} e^{(t - ig\varphi/N)\bar{\psi}\psi} = N! \int d\varphi (t - ig\varphi/N)^N e^{-g\varphi^2/2N} \quad (18)$$

When $N \rightarrow \infty$ the integral in (18) is determined by the saddle-point value of φ . Rescaling $\varphi \rightarrow \varphi/N$, this can be found from the stationary condition for the effective action

$$S(\varphi) \equiv -Ng\varphi^2/2 + N \log(t - ig\varphi) \quad (19)$$

appearing in (18) which is

$$tg\varphi - ig^2\varphi^2 + ig = 0 \quad (20)$$

The solution of (20) which is regular at $g = 0$ is

$$\varphi_0 = \frac{t}{2ig} \left(1 - \sqrt{1 - \frac{4g}{t^2}} \right) \quad (21)$$

Substituting (21) into (19) we get the tree-level fermionic free energy

$$F_F^{(0)} = \frac{1}{2} - \frac{t}{4g} \left(t - \sqrt{t^2 - 4g} \right) - \log \left(\frac{t}{2} + \sqrt{t^2 - 4g} \right) \quad (22)$$

Note that the saddle points in (10) and (21) are related by the correspondence $\phi^2 = \frac{i}{2}\varphi$ which makes more precise the analytical continuation discussed above. Notice also that the free energy (22) is related to the free energy (11) of the scalar model as $F_F^{(0)} = 1 - 2F_S^{(0)}$, as anticipated since the large- N limit of the 2 models represents the same combinatorial problem of enumerating tree-graphs.

The higher-loop contributions (which count the polymer networks with a given number of molecules) can be found by carrying out the saddle point calculation of the integral (18) to higher orders. For this, we decompose the Hubbard-Stratonovich field as $\varphi = \varphi_0 + \varphi_q$ and expand the action (19) in a Taylor series about the saddle-point value (21) in terms of the fluctuation fields φ_q . Using the saddle-point equation (20) when evaluating the higher-order derivatives $S^{(n)}(\varphi_0)$, this Taylor series is found to be

$$S(\varphi) = S(\varphi_0) - \frac{N}{2} (g + g^2\varphi_0^2) \varphi_q^2 - N \sum_{n=3}^{\infty} \frac{(-g\varphi_0)^n}{n} \varphi_q^n \quad (23)$$

The genus 1 free energy is then obtained from the fluctuation determinant that arises from Gaussian integration over the quadratic part in φ_q of (23),

$$F_F^{(1)} = \frac{1}{2N} \log(g + g^2 \varphi_0^2) = \frac{1}{2N} \log\left(2g - \frac{t^2}{2} + \frac{t}{2} \sqrt{t^2 - 4g}\right) \quad (24)$$

which also agrees with the 1-loop free energy of the $O(N)$ vector model [9].

The $\frac{1}{N}$ -expansion of the fermionic free energy also becomes non-analytic at the critical point $g = g_c = t^2/4$. It exhibits the same critical behaviour as the ϕ^4 theory above and it therefore lies in the same universality class as this statistical model. It is straightforward to carry out the double-scaling limit of the fermionic vector model in much the same way as in the bosonic case [6, 7, 9]. This limit is associated with the continuum limit of the polymer network at $N \rightarrow \infty$, $g \rightarrow g_c$ in such a way that a coherent contribution from all orders of the perturbative and $1/N$ expansions is obtained. We approach the critical point g_c by defining a dimensionless “lattice spacing” a by $at^2 = g_c - g$ and taking the continuum limit $a \rightarrow 0$. With the rescalings mentioned above, the contribution to an arbitrary ℓ -loop vacuum diagram with $\ell \geq 1$ is $N^{-b}(\sqrt{a})^{-b}N^n = N^{1-\ell}(\sqrt{a})^{1-\ell-n}$. As shown in [7], the maximum number of vertices that a 4-point polymer diagram with ℓ loops can have is $n = 2(\ell - 1)$, so that the most singular behaviour of an ℓ -loop diagram in the continuum limit $a \rightarrow 0$ is $(Na^{3/2})^{1-\ell}$. The proper continuum limit wherein a finite contribution from arbitrary genus polymers is obtained is thus the “double-scaling” limit where $N \rightarrow \infty$ and $a \rightarrow 0$ in a correlated fashion so that the renormalized “cosmological constant” (or “linear string tension”) $\Lambda \equiv Na^{3/2}$ remains finite. The double scaling limit enables an explicit construction of the genus expansion of the continuum polymer theory from the vector model.

We can now take the double scaling limit of the partition function (12) and write it as a loop expansion in the linear string tension Λ . However, as noted for the $O(N)$ vector model [7], the tree and 1-loop contributions are singular in this limit. The saddle-point value (21) can be written in terms of the lattice spacing as $g\varphi_0 = (2/it)(1 - 2\sqrt{a})$, from which it follows that the genus 0 and 1 free energies (22) and (24) are given by

$$NF_F^{(0)} \sim -\frac{N}{2} - N \log\left(\frac{t}{2}\right) + 6N^{1/3}\Lambda^{2/3} - \frac{32}{3}\Lambda \quad , \quad NF_F^{(1)} \sim \frac{1}{2} \log\left(\frac{t^2}{2}\right) + \frac{1}{2} \log\left(\frac{2\Lambda^{1/3}}{N^{1/3}}\right) \quad (25)$$

in the continuum limit $a \rightarrow 0$. The Λ -dependent terms in (25) diverge in the double scaling limit and represent a non-universal behaviour of the statistical polymer system. The tree-level and one-loop order Feynman diagrams should therefore be subtracted in the definition of the double-scaling limit leading to a renormalized partition function $Z_R(\Lambda, t)$ that only contains contributions from the ℓ -loop diagrams with $\ell \geq 2$.

This renormalized partition function is obtained by integrating over that part of the action involving $n \geq 3$ vertices in the fluctuation field φ_q weighted against the Gaussian form in (23).

To pick out the finite contribution in the double-scaling limit, we rescale the fluctuation field as $\varphi_q \rightarrow 2N^{1/3}\Lambda^{1/6}g_c^{-1/2}\varphi_q$ and note that with this rescaling we have

$$-\frac{N}{2}\left(g + g^2\varphi_0^2\right)\varphi_q^2 \rightarrow -\frac{1}{2}\varphi_q^2 \quad , \quad -N(-g\varphi_0)^n\varphi_q^n \rightarrow -\frac{N(g_c)^{n/2}}{2^n}\left(\frac{it}{2}\right)^n N^{-n/3}\Lambda^{-n/6}\varphi_q^n \quad (26)$$

in the continuum limit $a \rightarrow 0$. The $n \geq 4$ vertex terms in (26) vanish in the double scaling limit, and therefore the exact renormalized partition function in the double scaling limit is (up to irrelevant normalization factors)

$$Z_R(\Lambda, t) = \frac{\int d\varphi_q e^{-\frac{1}{2}\varphi_q^2 + \frac{it}{512}\Lambda^{-1/2}\varphi_q^3}}{\int d\varphi_q e^{-\frac{1}{2}\varphi_q^2}} = \sum_{k=0}^{\infty} \frac{i^k t^{6k}}{k!(512)^k} \Lambda^{-k/2} \left(\langle \phi^{3k} \rangle_S \Big|_{N=1, t=\frac{1}{2}} \right) \quad (27)$$

The Gaussian moments in (27) can be evaluated as in (6). The odd moments vanish, while the even moments yield a factor $(3k-1)!!$. Thus the double-scaled renormalized partition function admits the exact genus expansion

$$Z_R(\Lambda, t) = \sum_{\ell=0}^{\infty} (-1)^\ell t^{12\ell} \frac{(6\ell-1)!!}{(2\ell)!(512)^{2\ell}} \Lambda^{-\ell} \quad (28)$$

The partition function (28) has a similar structure as that in the $O(N)$ vector model where the genus expansion is an asymptotic series with zero radius of convergence [7]. In the fermionic case, however, the genus expansion is an alternating sum, and is therefore Borel summable. The convergence of the sum over genera is easily seen in the integral expression (27) where the unbounded cubic term contains a factor of i which makes the overall integration there finite. The Borel summability of the genus expansion is a feature unique to the fermionic models that does not usually occur for random geometry theories. In this sense, the fermionic vector model represents some novel discretized surface theory in which the topological expansion uniquely specifies the generating function of the statistical theory.

The two functions

$$Z_{\pm}(t, g; N) = \sqrt{Z_S(t, g; N)Z_F(t, g; N/2)^{\pm 1}} \quad (29)$$

are partition functions of random polymer models where, in the case of the $+$ sign the genera are restricted to be even and in the case of the $-$ sign the genera are odd. The resulting statistical theory is not Borel summable, but it does represent a new sort of generating function for “reduced” polymer systems. In these reduced statistical models, at least part of the non-universal genus 0 or 1 double-scaling behaviour is removed. The convergence properties of the fermionic vector models can therefore be exploited to combine them with other vector models and generate unusual statistical models of random polymer systems. It would be interesting to give the fermionic nature of these models an interpretation in terms of random surfaces directly. The identification $-\frac{1}{N} = e^\mu$ in the fermionic case suggests that

the associated random polymer theory has a complex-valued “fugacity” $\mu = i\pi + \mu_0$, $\mu_0 \in \mathbb{R}$, with doubly-degenerate degrees of freedom at each vertex. Heuristically, the complex fugacity μ can be thought of as a rotation of the real-valued one μ_0 required to compensate the extra vertex degree of freedom that does not appear in the bosonic models. From an analytic point of view, the genus sum alternates relative to that of the $O(N)$ vector model because the saddle-point (21) is imaginary in the fermionic vector model (12) so that its saddle-point expansion is the analytical continuation $\phi_0^2 = \frac{i}{2}\varphi_0$ of that for the scalar model.

The above analysis can be straightforwardly generalized to an interaction of the form $g(\bar{\psi}\psi)^K$, which will then represent a random polymer model with up to $2K$ -valence vertices. The critical behaviour is the same as that in a ϕ^{2K} scalar vector model and leads to the same susceptibility exponent $\gamma^{(0)} = \frac{1}{2}$, i.e. such a theory of random polymers is universal. To generate more complicated polymer models, for instance those with matter degrees of freedom at the vertices of the discretization [1], one must study vector models with more complicated interactions. One particularly interesting aspect of the fermionic vector model is the extent to which these more complex models can be solved explicitly. Consider the more general fermionic vector model with partition function

$$Z_0 = \int d\psi d\bar{\psi} e^{NV(\bar{\psi}\psi)} \quad (30)$$

where $V(z)$ is some “potential” function. The integration in (30) is well-defined and finite if $e^{NV(z)}$ has a well-defined Taylor expansion to order N in the variable z . In that case, the partition function (30) at finite N can be formally evaluated by inserting the delta function $1 = \int dz \delta(z - \bar{\psi}\psi)$ and using the identity

$$\int \frac{dw}{2\pi} \int d\psi d\bar{\psi} e^{iw(z - \bar{\psi}\psi)} = \int \frac{dw}{2\pi} N!(-iw)^N e^{i wz} = N! \left(-\frac{\partial}{\partial z}\right)^N \delta(z) \quad (31)$$

to obtain

$$Z_0 = N! \cdot \left(\frac{\partial}{\partial z}\right)^N e^{NV(z)} \Big|_{z=0} \quad (32)$$

Similarly, the correlators for N finite are given by

$$\langle (\bar{\psi}\psi)^n \rangle \equiv \frac{\int d\psi d\bar{\psi} (\bar{\psi}\psi)^n e^{NV(\bar{\psi}\psi)}}{\int d\psi d\bar{\psi} e^{NV(\bar{\psi}\psi)}} = \frac{\left(\frac{\partial}{\partial z}\right)^N z^n e^{NV(z)}}{\left(\frac{\partial}{\partial z}\right)^N e^{NV(z)}} \Big|_{z=0} \quad (33)$$

The generating function and observables for these fermionic polymer models can therefore always be explicitly and uniquely specified. It is straightforward to check that (32) reduces to (17) when the potential is the four-Fermi interaction that we studied above.

To treat the model (30) at $N = \infty$, we use the first part of the identity (31) to write the partition function as

$$Z_0 = \frac{(-i)^N N!}{2\pi} \int dz dw e^{NV(z) + i wz + N \log w} \quad (34)$$

If the potential is a polynomial of degree m ,

$$V(z) = \sum_{k=1}^m g_k z^k \quad (35)$$

then we can rescale $z \rightarrow z/N$ and the coupling constants $g_k \rightarrow N^k \cdot g_k$ simultaneously so that the effective action in (34) is $NV(z) + iNwz + N \log w$. The integral (34) at large- N is determined by the saddle-point value of this effective action. In the 2-dimensional complex space of the variables w and z , the stationary conditions are

$$V'(z) + iw = 0 \quad , \quad iz + 1/w = 0 \quad (36)$$

which can be combined into the single equation

$$zV'(z) = 1 \quad (37)$$

The equation (37) is identical to the stationary condition for the $O(N)$ vector model defined with potential $V(\phi^2)$ [7, 8]. Thus the critical behaviour of the fermionic vector model (30) is the same as that for the $O(N)$ vector model with the same polynomial potential (35). The genus expansion is generated by the 2-dimensional saddle-point evaluation of the integral (34). The imaginary saddle-point values given by (36) will lead to an alternating genus expansion in the double-scaling limit for the fermionic theory, leading to a Borel summable polymer model, in contrast to the scalar case. In this case the critical point is again that point in coupling constant space where the function $zV'(z)$ vanishes. For a potential of the form (35), we can adjust the coupling constants in such a way that the critical point is a zero of $zV'(z)$ of order m . The leading singularity of the free energy will then be $a^{(m+1)/m}$ [6, 7] which leads to the critical susceptibility exponent $\gamma^{(0)} = 1 - 1/m$, $m = 2, 3, \dots$. This is the multicritical series for generalized random polymer systems in dimension $D \geq 0$ [1] which interpolates between the Cayley tree at $m = \infty$ with $\gamma^{(0)} = 1$ and the ordinary random walk we discussed earlier at $m = 2$ with $\gamma^{(0)} = \frac{1}{2}$ (the N^0 -component of the vector model free energy represents the self-avoiding random walk). In the $O(N)$ models, the former case would represent a phase of bosonic strings in target space dimension $D \geq 1$ while the latter case would represent a phase of pure 2-dimensional quantum gravity. In the general case, the potential (35) leads to discrete filamentary surfaces which have vertices of even valence up to $2m$. It would be interesting to determine what physical systems the fermionic vector models represent in the continuum limit.

We see therefore that the fermionic vector model partition function is a well-defined, finite generating function for polymer configurations which can always (for any N) be evaluated exactly. The nature of its genus expansion, i.e. that it has an alternating, Borel summable double scaling series, agrees with conjectures concerning the nature of the topological expansion in fermionic matrix models which have been studied in [11] as alternative random surface

theories to the conventional Hermitian matrix models. There it is conjectured that the genus expansion is an alternating series which may be Borel summable but otherwise coincides with the usual Painlevé expansion [3]. This is expected to be only true for odd polynomial potentials as it is only in that case that the matrix model possesses a chiral symmetry and imaginary endpoints for the support of the spectral distribution. In the case of the simpler fermionic vector models the partition function is always invariant under chiral transformation of the fermionic vector components. Furthermore, in the case of fermionic matrix models the correspondence with a scalar theory is more complicated – a polynomial fermion model can be analytically continued to a Hermitian matrix model with a generalized Penner potential which consists of a logarithmic plus polynomial potential [11]. It should therefore represent a Borel summable generating function for the virtual Euler characteristics of the discretized moduli spaces of Riemann surfaces (rather than just the generating function for a random surface triangulation itself). In the vector case, the fermionic model represents the same type of random surface theory as the corresponding $O(N)$ vector field theory. Nevertheless, further study of the more technical features of fermionic vector models, such as an analysis of the Schwinger-Dyson (loop) equations and the connection with integrable hierarchies, may help in understanding what the corresponding structures will look like in the case of fermionic matrix models. In the vector case, they will generate examples of fermionic models which are explicitly solvable.

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